solution, having jumps at $\omega^d t = \pi / 2 + \pi m$ (Fig. 1, curve y_c). For frequencies smaller than ω_d, the weak discontinuities of this solution disappear and the solution is described by formula (3.2) for y_b.

Thus, the periodic solution of Eq. (1.7) with the boundary conditions (1.5) is unique for $\omega > \omega^c$; there are two solutions for $\omega^c \geqslant \omega > \omega^d$ (one is discontinuous, the other is continuous); three solutions (one continuous, two discontinuous) are possible for $\omega < \omega^a$

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ON THE PRESSURE ON AN ELASTIC HALF-SPACE BY A WEDGE-SHAPED STAMP

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The problem of the effect of an absolutely rigid stamp with a wedge planform on an elastic space is considered. There is assumed to be no friction in the domain of contact between the stamp and the half-space.

Galin first considered this problem in [1]. The effect of the stamp on the half-space was accompanied, in that paper, by the effect of some loading outside it. A characteristic singularity of this solution is the fact that the contact pressures $p(\tau, \varphi)$ have a r^{-1} singularity at the wedge apex.

Later, Rvachev attempted to solve the mentioned problem without the outside loading [2]. He reduced it to an eigenvalue problem for a certain differential equation on a sphere and utilized the Galerkin method. The Rvachev solution has a $r^{\gamma-1}$ singularity at the wedge apex, where $0 < \gamma(\alpha) < 1$, and 2α is the wedge angle.

In this paper the problem of a wedge-shaped stamp with an arbitrary base is apparently successfully solved analytically for the first time by utilizing the asymptotic "method of large λ" [3], and the singularity in the contact pressure at the wedge apex is isolated exactly for sufficiently small α. It hence turns out that in the general case the function $p(r, \varphi)$ behaves as $r^{-3/2}\cos(\theta \ln r)$ in the neighborhood of the point $r = 0$, where $\theta = \theta(\alpha)$. The r^{-1} and $r^{\gamma-1}$ singularities also hold, but are contained in the following members of the asymptotic of the function $p(r, \varphi)$ as $r \to 0$.

The question of the construction of an asymptotic solution of the problem under consideration for wedge angles near 2π is also posed herein.

1. As is known [4], the problem of the impression of a stamp with a wedge planform reduces to the solution of an integral equation of the form

$$\int_{-\alpha}^{\alpha} d\psi \int_{0}^{\infty} \frac{p(\rho, \psi)\rho d\rho}{\sqrt{\rho^2 + r^2 - 2\rho r\cos(\varphi - \psi)}} = 2\pi\Delta f(r, \varphi) \qquad \begin{pmatrix} 0 \leqslant r < \infty \\ |\varphi| \leqslant \alpha \end{pmatrix} \tag{1.1}$$

Here 2α is the wedge angle, $f(r, \varphi)$ is a function defined by the shape of the stamp base and the degree of its insertion into the half-space, $p(\rho, \psi)$ is the contact stress under the stamp, while G and ν are the shear modulus and Poisson's ratio of the half-space material, respectively.

Let us note that the formulation of the problem of a plane wedgelike stamp, and precisely this case has been examined in [1, 2], is not completely correct because for $f(r, \varphi) \equiv f = \text{const}$ only a solution with infinite energy can exist. Taking this into account, let us henceforth consider only the case when a Mellin [5] transformation in the variable r is applicable to the function $f(r, \varphi)$ and

$$\int_{-\alpha}^{\alpha} d\psi \int_{0}^{\infty} |f(\rho, \psi)|\rho\, d\rho < \infty \tag{1.2}$$

We shall require compliance with these same conditions from the solution $p(r, \varphi)$ also.

Now applying the Mellin transform in r to both sides of (1.1), we obtain [5]

$$\int_{-1}^{1} p_s(\xi) K_s\left(\frac{\xi - x}{\lambda}\right) d\xi = \pi f_s(x) \qquad (|x| \leqslant 1) \tag{1.3}$$

$$p_s(\xi) = p_s^*(\psi), \ f_s(x) = \alpha^{-1} f_s^*(\varphi), \ \xi = \psi\alpha^{-1}, \ x = \varphi\alpha^{-1}, \ \lambda = \alpha^{-1} \tag{1.4}$$

$$p_s^*(\psi) = \int_0^\infty p(\rho, \psi)\rho^{s+1/2} d\rho, \ p(\rho, \psi) = \frac{1}{2\pi i}\int_\Gamma p_s^*(\psi)\rho^{-s-3/2} ds \tag{1.5}$$

$$f_s^*(\varphi) = \Delta\int_0^\infty f(r, \varphi) r^{s-1/2} dr, \ \Delta f(r, \varphi) = \frac{1}{2\pi i}\int_\Gamma f_s^*(\varphi) r^{-s-1/2} ds \tag{1.6}$$

$$K_s(\theta) = \frac{1}{2}\int_0^\infty \frac{t^{s-1/2}dt}{\sqrt{1 + 2t\cos(\pi - \theta) + t^2}} = \frac{\pi}{2\cos\pi s} P_{s-1/2}(-\cos\theta) \begin{pmatrix} |\operatorname{Re} s| < 1/2 \\ \theta = (\xi - x)/\lambda \end{pmatrix} \tag{1.7}$$

Here Γ is a line in the plane of the complex variable $s = \sigma + i\tau$, parallel to the imaginary axis and $P_\nu(x)$ is the Legendre function on the slit [6]. The following asymptotic expansion [6] holds for the kernel (1.7):

$$K_s(\theta) = \frac{\pi}{2\cos\pi s} F\left(\frac{1}{2}+s,\ \frac{1}{2}-s,\ 1,\ \cos^2\frac{\theta}{2}\right) \tag{1.8}$$

$$F\left(\frac{1}{2}+s,\ \frac{1}{2}-s,\ 1,\ \cos^2\frac{\theta}{2}\right) = \left(\frac{\cos\pi s}{\pi}\right)^2 \sum_{n=0}^{\infty} \frac{\Gamma(1/_2+s+n)\,\Gamma(1/_2-s+n)}{(n!)^2} \times$$

$$\times\left(h_n - 2\ln\sin\frac{\theta}{2}\right)\sin^2\frac{n\theta}{2},\quad h_n = 2\psi(n+1) - \psi(n+1/_2+s) - \psi(n+1/_2-s,$$

$$\psi(z) = \Gamma'(z)\,/\,\Gamma(z)$$

We finally obtain for $K_s(\theta)$

$$K_s(\theta) = -\ln|\theta| + a_0 + \sum_{i=1}^{\infty}(a_i + b_i\ln|\theta|)\,\theta^{2i} \tag{1.9}$$

where the expansion (1.9) converges uniformly for all $|\theta| \leqslant \pi - \varepsilon,\ \varepsilon > 0$. Some of the first coefficients of the series (1.9) are

$$a_0 = \psi(1) - 0.5\psi(0.5+s) - 0{,}5\psi(0.5-s) + \ln 2$$

$$a_1 = 0.125\,(0.25 - s^2)\times$$

$$\times[2\psi(2) - \psi(1.5+s) - \psi(1.5-s) + 2\ln 2] + 0.041\,(6)$$

$$a_2 = 0.000347\,(2) + 0.0078125\,(0.25-s^2)\,(2.25-s^2)\times$$

$$\times\ [2\psi(3) - \psi(2.5+s) - \tag{1.10}$$

$$-\psi(2.5-s) + 2\ln 2] + 0.01041\,(6)\,(0.25-s^2)\,[1 - 2\psi(2) + \psi(1.5+s) +$$

$$+\psi(1.5-s) - 2\ln 2],\ b_1 = -0.25\,(0.25-s^2)$$

$$b_2 = 0.0208\,(3)\,(025 - s^2) - 0.015625\,(0.25-s^2)\,(2.25-s^2)$$

The asymptotic solution of the integral equation (1.3) with kernel (1.9) can be obtained for small α by the method of large λ [3]. Furthermore, for definiteness let us limit ourselves to the case

$$f(r,\varphi) = fr^{\mu}e^{-\varkappa r} \qquad (\mu \geqslant \delta - 1,\ \delta > 0,\ \varkappa > 0) \tag{1.11}$$

For $\mu = 0$ and $\varkappa \to$ such a stamp degenerates into a plane one. Utilizing the tables [5] we find

$$f_s(x) = \Delta f\alpha^{-1}\varkappa^{-(s+1/_2+\mu)}\,\Gamma(s+1/_2+\mu),\quad \operatorname{Re}s > -0.5-\mu \tag{1.12}$$

For the case $f_s(x) \equiv \gamma(s)$, the asymptotic solution of (1.3), (1.9) has been obtained with accuracy in [3] to terms of $\lambda^{-6}\ln^3\lambda$ and is determined by (12) - (14). Taking this into account we will have for the function $p_s^*(\varphi)$

$$p_s^*(\varphi) = (\alpha^2-\varphi^2)^{-1/_2}D^{-1}(s,\lambda)\sum_{m=0}^{3}\lambda^{-2m}\sum_{k=0}^{m}[c_{mk}(s) + d_{mk}(s)\ln\lambda]\,(\varphi/\alpha)^{2k} +$$

$$+O(\lambda^{-6}\ln^3\lambda),\quad c_{00}(s) = \Delta f\varkappa^{-(s+1/_2+\mu)}\Gamma(s+1/_2+\mu),\quad d_{00}(s) = 0 \tag{1.13}$$

$$D\ (s,\lambda) = a_0 + \ln 2\lambda + (\delta_1 - b_1\ln 2\lambda)\lambda^{-2} + (\delta_2 + \delta_s\ln 2\lambda -$$

$$-0.25b_1^2\ln^2 2\lambda)\lambda^{-4} + O(\lambda^{-6}\ln^3\lambda),\ \delta_1 = a_1 + b_1,\ \delta_2 = -0.25a_1^2 - 0.75a_1b_1 -$$

$$-0.5625b_1^2 + 2.25a_2 + 2.625b_2,\quad \delta_s = 0.5a_1b_1 + 0.75b_1^2 - 2.25\,b_2$$

The expressions for the remaining coefficients $c_{mk}(s)$ and $d_{mk}(s)$ are not needed later.

An approximate solution of the problem can be obtained for small α by means of (1.5) by using residue theory, however, only the zeroes of the function $D(s,\lambda)$ are known. The location of the line Γ is selected from the conditions of convergence of the first integral in (1.5) and absolute integrability of the function $p(\rho, \psi)$ over the wedge domain.

2. Let α be so small that terms of order λ^{-2} and higher can be neglected in (1.13) (1.14) with sufficient accuracy. Then in conformity with the second formula of (1.5), the solution of the problem can be represented as

$$p(r, \varphi) = \frac{\Delta f \varkappa^{-\mu+1}}{2\pi i \sqrt{\alpha^2 - \varphi^2}} \int_\Gamma \frac{(r\varkappa)^{-s-3/2}\Gamma(s + 1/2 + \mu)\, ds}{\ln 4\lambda - C - 1/2\psi(1/2 + s) - 1/2\psi(1/2 - s)} \tag{2.1}$$

where C is the Euler constant.

Let us study the zeroes of the function

$$g(s) = \psi(1/2 + s) + \psi(1/2 - s) - \varkappa, \quad \varkappa = 2\ln 4\lambda - 2C \tag{2.2}$$

in the strip $|\operatorname{Re} s| \leqslant 3/2$

Theorem. For $\lambda > 1$ the function $g(s)$ has only four single zeroes in the strip $|\operatorname{Re} s| \leqslant 3/2$ (2.3)

$$s_{1,2} = \pm[1/2 + \gamma(\alpha)], \ \gamma(\alpha) = O(\varkappa^{-1}) > 0, \ s_{3,4} = \pm i\theta(\alpha), \ \theta(\alpha) = O(\alpha^{-1})$$

For the proof, let us note that

$$g(3/2 - 0) = -\infty; \ g(1/2 \pm 0) = \pm\infty, \ g(0) < 0, \ g(i\infty) = +\infty \tag{2.4}$$

Asymptotic formulas can be established for $\gamma(\alpha)$ and $\theta(\alpha)$ if the representation (6) ([6], sec. 1.7) is used for the function $\psi(s)$. The uniqueness and simpleness of the zeroes s_k $(k = 1, 2, 3, 4)$ can be proved if the principle of the argument is utilized. It must just be taken into account that the function $g(s)$ has two single poles at the points $s = \pm 1/2$ in the rectangle $|\sigma| \leqslant 3/2 - 0$, $|\tau| \leqslant A < \infty$.

Let Γ be the straight line

$$\sigma = 1/2 - \varepsilon, -\infty < \tau < \infty, \ 0 < \varepsilon < \operatorname{Inf}(1/2, \ 1 + \mu).$$

We then find by using residue theory that as $r \to 0$

$$p(r, \varphi) = \frac{\Delta f}{\sqrt{\alpha^2 - \varphi^2}} \{r^{\mu-1}A(\mu) - \varkappa r^{\mu}A(\mu + 1) + \varkappa^{\gamma-\mu}r^{\gamma-1}B(\mu, \gamma) -$$

$$- \varkappa^{-\mu-1/2}r^{-3/2}[\cos(\theta \ln r\varkappa)C(\mu, \theta) - \sin(\theta \ln r\varkappa)D^*(\mu, \theta)] + \varkappa^{1-\mu}o(1)\}$$

$$A(\mu) = \frac{1}{\ln 4\lambda - C - 1/2\psi(-\mu) - 1/2\psi(1 + \mu)}, \quad B(\mu, \gamma) = \frac{2\Gamma(\mu - \gamma)}{\psi'(1 + \gamma) - \psi'(-\gamma)}$$

$$C(\mu, \theta) = 2\operatorname{Im}\left[\frac{\Gamma(1/2 + \mu + i\theta)}{\psi'(1/2 + i\theta)}\right], \quad D^*(\mu, \theta) = 2\frac{\operatorname{Re}\Gamma(1/2 + \mu + i\theta)}{\operatorname{Im}\psi'(1/2 + i\theta)} \tag{2.5}$$

We find analogously that as $r \to \infty$

$$p(r, \varphi) = \frac{\Delta f}{\sqrt{\alpha^2 - \varphi^2}}\left\{- r^{-2-\gamma}\varkappa^{-1-\mu-\gamma}\frac{2\Gamma(1 + \gamma + \mu)}{[\psi'(1 + \gamma) - \psi'(-\gamma)]} + \varkappa^{-\mu-2}O(r^{-3})\right\} \tag{2.6}$$

It follows from (2.5) that in the worst case as $r \to 0$

$$p(r, \varphi) \sim O(r^{\delta-2}) \tag{2.7}$$

Substituting (2.6), (2.7) into the first formula of (1.5), we see that the integral converges if

$$^1/_2 - \delta < \operatorname{Re} s < {}^1/_2 + \gamma \tag{2.8}$$

We also see that the function $p\ (r,\ \varphi)$ is absolutely summable over the wedge domain. The line Γ is located in such a way that all the constraints imposed on $\operatorname{Re} s$ are simultaneously satisfied.

An investigation of the solution obtained shows that if $\delta - 1 \leqslant \mu < -{}^1/_2$, then the principal singularity of the function $p\ (r,\ \varphi)$ in the neighborhood of $r = 0$ is $r^{\mu-1}$, the oscillating singularities $r^{-3/2} \cos (\theta \ln r\varkappa)$ and $r^{-3/2} \sin\ (\theta \ln r\varkappa)$ are secondary, and the singularity r^{μ} or $r^{\gamma-1}$ is ternary. If $-{}^1/_2 \leqslant \mu < \gamma$, then the oscillating singularities become the principal singularities. Consequently, the contact pressure $p\ (r,\ \varphi)$ will change sign an infinite number of times as $r \to 0$, as holds in contact problems with total adhesion. Therefore, an elastic medium can already not adjoin the stamp surface compactly in the neighborhood of the wedge apex. The singularity $r^{\mu-1}$ turns out to be secondary, and $r^{\gamma-1}$ or r^{μ} ternary. Finally, if $\gamma \leqslant \mu$, then, as before, the oscillatory singularities will be the principal singularities, but the singularity $r^{\gamma-1}$ becomes secondary, and $r^{\mu-1}$ ternary.

By using the Rouchet theorem it can be shown that the qualitative picture described does not change if all the components mentioned are retained in (1.13).

3. It is interesting to construct the asymptotics of the solution of the problem for small $\beta = \pi - \alpha$. It is here necessary to turn attention to the fact that the kernel (1.7) of the integral equation (1.3) has period 2π. A similar kind of equation has been studied in [7], where it was reduced to some infinite algebraic system, specified best for small β. An analysis of this system would permit construction of the principal term of the asymptotics of the solution in the form (11) [7].

We shall give here another method of reducing an integral equation of the type (1.3), (1.7) to an infinite algebraic system.

Let the integral equation

$$\int_{-\pi+\beta}^{\pi-\beta} q\ (\psi)\ K\ (\varphi - \psi)\ d\psi = 2\pi\ f\ (\varphi) \qquad (|\ \varphi\ | \leqslant \pi - \beta) \tag{3.1}$$

whose kernel $K\ (t)$ has the period 2π. We rewrite (3.1) in the form

$$\int_{-\pi}^{\pi} q^*\ (\varphi)\ K\ (\varphi - \psi)\ d\psi = 2\pi f\ (\varphi) \qquad (|\ \varphi\ | \leqslant \pi - \beta)$$

$$q^*\ (\psi) = q\ (\psi) \quad \text{for } |\ \psi\ | \leqslant \pi - \beta,\ q^*\ (\psi) = 0 \quad \text{for } \ \pi - \beta < |\ \psi\ | \leqslant \pi \tag{3.2}$$

and expand the functions $q^*\ (\psi)$, $K\ (\theta)$ and $f\ (\varphi)$ in Fourier series

$$K\ (\theta) = \sum_{m=-\infty}^{\infty} k_m e^{im\theta}, \quad q^*\ (\psi) = \sum_{n=-\infty}^{\infty} q_n e^{in\psi}, \quad f\ (\varphi) = \sum_{s=-\infty}^{\infty} f_s \exp\left(\frac{i\pi s\varphi}{\pi - \beta}\right) \tag{3.3}$$

Substituting the expansions (3.3) into (3.2) and integrating, we obtain

$$\sum_{n=-\infty}^{\infty} q_n k_n e^{\ n\varphi} = \sum_{s=-\infty}^{\infty} f_s \exp\left(\frac{i\pi s\varphi}{\pi - \beta}\right) \tag{3.4}$$

Taking account of the formula

$$e^{in\varphi} = \sum_{s=-\infty}^{\infty} \frac{\sin\left(\pi n - \beta n - \pi s\right)}{\pi n - \beta n - \pi s} \exp\left(\frac{i\pi s\varphi}{\pi - \beta}\right) \tag{3.5}$$

we obtain the following infinite algebraic system in the unknown coefficients q_n :

$$\sum_{n=-\infty}^{\infty} q_n k_n \frac{\sin(\pi n - \beta n - \pi s)}{\pi n - \beta n - \pi s} = f_s \qquad (s = \ldots -2, -1, 0, 1, 2, \ldots) \qquad (3.6)$$

For sufficiently small β the system (3.6) can be written as

$$q_s k_s - \frac{\beta}{\pi} \sum_{n=-\infty}^{\infty}{}' \frac{n q_n k_n}{n - s} (-1)^{n-s} = f_s \qquad (s = \ldots -1, 0, 1, \ldots) \qquad (3.7)$$

where the prime on the summation sign means that the member corresponding to $n = s$ has been omitted. An approximate solution of (3.7) for small β can be obtained by successive approximations.

It is also interesting to construct the asymptotics of the solution of the problem for small $\eta = \pm(\pi / 2 - \alpha)$. In this case it should be noted that the solution of the integral equation (1.1) for $\alpha = \pi/2$ (the stamp is a half-plane in planform) can be found in closed form. It is natural to take it as the zero approximation.

Let us note that (1.5), (1.13), (2.1), (2.5), (2.6) also solve the problem of impression of a stamp outlined by the arcs of two circles intersecting at a small angle $2d$, on an elastic half-space. It is only necessary to subject them to a Kelvin inversion transformation.

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